Variational Formulation for Two-Fluid Plasmas in Clebsch Variables

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A variational principle for two-fluid plasmas is obtained by using Clebsch variables. A Hamiltonian formalism is derived and canonical Poisson brackets are defined.

1. Introduction

There is interest in developing variational principles in fluid and plasma dynamics within the Eulerian description to study nonlinear fluctuations [1]. This description does not preserve a close similarity with a system of particles as the Lagrangian one does. The main difficulty is that the set of equations in the form originally given does not follow from a variational principle. An equivalent system which folllows from a variational principle has to be found through transformations and different representations of the variables.

Clebsch [2] introduced Euler-like potential variables for the fluid velocity to derive the hydrodynamic equations for an ideal and incompressible fluid from a variational principle. Several studies have meanwhile appeared on the subject. Seliger and Whitham [3] derived a variational principle for the equations of plasmas described by the two-fluid model, neglecting thermal motion. They used a combination of the potential representation for Maxwell's equations with the Clebsch potentials for the fluid equations.

With the internal energy per unit mass as a function of the density and entropy, the scheme proposed in [3] is now extended to two-fluid plasmas with finite pressure. A Hamiltonian description is introduced so that canonical Poisson brackets can be defined in terms of the (non-physical) fields. Gravitating two-fluid plasmas are also considered.

The difficulties in applying the same scheme when finite gyroradius effects are considered are also discussed. This suggests the necessity of a new representation to find a variational principle for the set of equations which describe this problem.

2. Clebsch representation and variational principle for two-fluid plasmas

The Eulerian equations for the two-fluid model are

$$m_s n_s \frac{\mathrm{d}\boldsymbol{v}_s}{\mathrm{d}t} = -\nabla P_s + e_s(\boldsymbol{E} + \boldsymbol{v}_s \times \boldsymbol{B}), \quad (1)$$

$$\dot{n}_s = \nabla \cdot (n_s \boldsymbol{v}_s) = 0, \quad (2)$$

$$\dot{\boldsymbol{B}} + \nabla \times \boldsymbol{E} = 0, \quad (3)$$

$$\varepsilon_0 c^2 \nabla \times \boldsymbol{B} = \sum_s e_s n_s \boldsymbol{v}_s + \varepsilon_0 \dot{\boldsymbol{E}}, \quad (4)$$

$$\varepsilon_0 \nabla \cdot \boldsymbol{E} = \sum_s e_s n_s, \quad (5)$$

$$\nabla \cdot \boldsymbol{B} = 0, \quad (6)$$

$$\mathrm{d}S_s/\mathrm{d}t = 0, \quad (7)$$

where m_s, e_s are the mass and charge of each particle of species s and n_s is the number density, $\boldsymbol{v}_s$ the velocity, P_s the pressure, S_s entropy per unit mass, $\boldsymbol{E}$ the electric field, and $\boldsymbol{B}$ the magnetic induction.

The internal energy per unit mass, U_s, and P_s are taken as functions of n_s and S_s related to the other thermodynamic quantities by

$$\mathrm{d}U_s = T_s\,\mathrm{d}S_s - P_s\,\mathrm{d}(m_s n_s)^{-1}, \quad (8)$$

where $T_s(n_s, S_s)$ is the temperature.

The combined representation

$$\boldsymbol{E} = -\dot{\boldsymbol{A}} - \nabla\chi,$$
$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \quad (9)$$
$$m_s \boldsymbol{v} + e_s \boldsymbol{A} = \nabla \Phi_s + \alpha_s \nabla \beta_s + S_s \nabla \eta_s$$

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is introduced. Eq. (1) can be written as

$$\nabla(\dot{\Phi}_s + \alpha_s \dot{\beta}_s + \dot{S}_s \dot{\eta}_s + e_s \chi + \tfrac{1}{2} m_s v_s^2 + m_s H_s) - \nabla \alpha_s \left(\frac{\mathrm{d}\beta_s}{\mathrm{d}t}\right) + \nabla \beta_s \left(\frac{\mathrm{d}\alpha_s}{\mathrm{d}t}\right) + \nabla \eta_s \left(\frac{\mathrm{d}S_s}{\mathrm{d}t}\right) - \nabla S_s \left(\frac{\mathrm{d}\eta_s}{\mathrm{d}t} + m_s T_s\right) = 0\,, \quad (10)$$

where

$$H_s = U_s + P_s/(m_s, n_s) \quad (11)$$

denotes the enthalpy. Then, using (7) and imposing the condition

$$\mathrm{d}n_s/\mathrm{d}t + m_s T_s = 0\,, \quad (12)$$

one can reduce (10) to

$$\dot{\Phi}_s + \alpha_s \dot{\beta}_s + S_s \dot{\eta}_s + e_s \chi + \tfrac{1}{2} m_s v_s^2 + m_s H_s = F_s(\alpha_s, \beta_s, t)\,, \quad (13)$$

$$\mathrm{d}\alpha_s/\mathrm{d}t = \partial F_s/\partial \beta_s\,, \quad (14)$$

$$\mathrm{d}\beta_s/\mathrm{d}t = -\,\partial F_s/\partial \alpha_s\,, \quad (15)$$

where $F_s(\alpha_s, \beta_s, t)$ are arbitrary functions. Here α_s and β_s are chosen such that

$$F_s = 0\,. \quad (16)$$

All solutions of (1) to (7) satisfy (9), (12), (13), (14) and (15). Clearly a solution of (2), (4), (5), (7), (9), (12), (13), (14) and (15) satisfy (1).

Equations (3) and (6) are satisfied identically by the representation (9) and the equations for Φ_s, α_s, β_s, η_s, S_s, χ, $\boldsymbol{A}$ are (2), (4), (5), (7), (12), (13), (14) and (15). These can be obtained, for the representation (9), from the variational principle

$$\delta \int\!\int \mathrm{d}\boldsymbol{r}\, \mathrm{d}t\, \mathscr{L} = 0\,, \quad (17)$$

where

$$\mathscr{L} = \varepsilon_0 \left(\frac{E^2}{2} - \frac{c^2}{2} B^2\right) - \sum_s n_s (\dot{\Phi}_s + \alpha_s \dot{\beta}_s + S_s \dot{\eta}_s + e_s \chi + \tfrac{1}{2} m_s v_s^2 + m_s U_s)\,. \quad (18)$$

The Lagrangian density can also be written as

$$\mathscr{L} = \varepsilon_0 \left(\frac{E^2}{2} - \frac{c^2}{2} B^2\right) + \sum_s P_s(m_s n_s, S_s) \quad (19)$$

by using (11) and (13).

3. Hamiltonian structure for two-fluid plasmas

From the given Lagrangian formulation it is possible to go over to a Hamiltonian formalism. One can define the momenta conjugate to the fields Φ_s, β_s, η_s and $\boldsymbol{A}$:

$$\pi_{\Phi_s} \equiv \frac{\partial \mathscr{L}}{\partial \dot{\Phi}_s} = -\,n_s\,, \quad (20)$$

$$\pi_{\beta_s} \equiv \frac{\partial \mathscr{L}}{\partial \dot{\beta}_s} - n_s \alpha_s\,, \quad (21)$$

$$\pi_{\eta_s} \equiv \frac{\partial \mathscr{L}}{\partial \dot{\eta}_s} = -\,n_s S_s\,, \quad (22)$$

$$\pi_{A_j} \equiv \frac{\partial \mathscr{L}}{\partial \dot{A}_j} = -\,\varepsilon_0 E_j\,; \quad j = 1, 2, 3\,. \quad (23)$$

The Hamiltonian density given by

$$\mathscr{H} = \sum_s (\dot{\Phi}_s \pi_{\Phi_s} + \dot{\beta}_s \pi_{\beta_s} + \dot{\eta}_s \pi_{\eta_s}) + \sum_j \dot{A}_j \pi_{A_j} - \mathscr{L} \quad (24)$$

becomes

$$\mathscr{H} = \sum_s \left[\frac{-1}{2 m_s \pi_{\Phi_s}} (\pi_{\Phi_s} \nabla \Phi_s + \pi_{\beta_s} \nabla \beta_s + \pi_{\eta_s} \nabla \eta_s - e_s \pi_{\Phi_s} \boldsymbol{A})^2 - m_s \pi_{\Phi_s} U_s(\pi_{\Phi_s}, \pi_{\eta_s}) - e_s \pi_{\Phi_s} \chi\right] + \varepsilon_0 \left[\frac{c^2}{2} (\boldsymbol{\nabla} \times \boldsymbol{A})^2 + \frac{1}{2 \varepsilon_0^2} \sum_j \pi_{A_j}^2\right]. \quad (25)$$

The Hamiltonian can also be written as

$$H = \int \mathrm{d}\boldsymbol{r} \left[\varepsilon_0 \left(\frac{E^2}{2} + \frac{c^2}{2} B^2\right) + \sum_s m_s n_s \left(\frac{v_s^2}{2} + U_s\right)\right]. \quad (26)$$

The Hamiltonian density obtained represents the total energy density.

The Eqs. (2), (4), (5), (7), (12), (13), (14), and (15) can be derived by appropriately combining the Hamilton equations of motion:

$$\dot{\Phi}_s = \frac{\delta H}{\delta \pi_{\Phi_s}}, \quad \dot{\pi}_{\Phi_s} = -\frac{\delta H}{\delta \Phi_s},$$
$$\dot{\beta}_s = \frac{\delta H}{\delta \pi_{\beta_s}}, \quad \dot{\pi}_{\beta_s} = -\frac{\delta H}{\delta \beta_s},$$
$$\dot{\eta}_\sigma = \frac{\delta H}{\delta \pi_{\eta_s}}, \quad \dot{\pi}_{\eta_s} = -\frac{\delta H}{\delta_{\eta_s}},$$
$$\dot{A}_j = \frac{\delta H}{\delta \pi_{A_j}}, \quad \dot{\pi}_{A_j} = -\frac{\delta H}{\delta A_j}, \quad (27)$$

where the functional derivatives are defined in the standard way after adding the term $\nabla \cdot (\varepsilon_0 \chi \boldsymbol{E})$ to the integrand of (26).

Canonical Poisson brackets in terms of the Clebsch variables here used can be defined as

$$\{M, N\} \equiv \int \mathrm{d}\boldsymbol{r} \left\{ \sum \left[\frac{\delta M}{\delta \Phi_s} \frac{\delta N}{\delta \pi_{\Phi_s}} - \frac{\delta N}{\delta \pi_{\Phi_s}} \frac{\delta M}{\delta \Phi_s} + \frac{\delta M}{\delta \alpha_s} \frac{\delta N}{\delta \pi_{\alpha_s}} - \frac{\delta N}{\delta \pi_{\alpha_s}} \frac{\delta M}{\delta \alpha_s} + \frac{\delta M}{\delta S_s} \frac{\delta N}{\delta \pi_{S_s}} - \frac{\delta N}{\delta \pi_{S_s}} \frac{\delta M}{\delta s_s} \right] + \sum \left[\frac{\delta M}{\delta A_j} \frac{\delta N}{\delta \pi_{A_j}} - \frac{\delta N}{\delta \pi_{A_j}} \frac{\delta M}{\delta A_j} \right] \right\}, \quad (28)$$

where M and N are arbitrary functionals of the canonical fields.

The Hamiltonian structure for two-species fluid electrodynamics was recently [4] obtained in terms of physical variables but the corresponding Poisson brackets are not canonical.

4. Variational principle for gravitating two-fluid plasmas

It may be of interest to consider gravitational forces for problems in astrophysics (star formation for example) and possibly for geophysical applications.

Under these circumstances the equation of motion (1) becomes

$$m_s n_s \frac{\mathrm{d}\boldsymbol{v}_s}{\mathrm{d}t} = -\nabla P_s + e_s (\boldsymbol{E} + \boldsymbol{v}_s \times \boldsymbol{B}) + m_s n_s \boldsymbol{g} . \quad (29)$$

The vector $\boldsymbol{g}$ can be written as

$$\boldsymbol{g} = -\nabla \psi , \quad (30)$$

where

$$\Delta \psi = -4\pi G \sum_s n_s m_s . \quad (31)$$

By using the combined representation (9) one can reduce (29) to a system of equations similar to eqs. (14) and (15) and to equation

$$\dot{\Phi}_s + \alpha_s \dot{\beta}_s + S_s \dot{\eta}_s + e_s \chi + \tfrac{1}{2} m_s v_s^2 + m_s H_s + m_s \psi = 0 . \quad (32)$$

The equations for Φ_s, α_s, β_s, η_s, S_s, χ and $\boldsymbol{A}$ can be obtained, for the representation used, from a constrained variational principle where Ψ has to obey (31). The Lagrangian density is

$$\mathscr{L} = \varepsilon_0 \left(\frac{E^2}{2} - \frac{c^2}{2} B^2 \right) - \sum_s \eta_s (\dot{\Phi}_s + \alpha_s \dot{\beta}_s + S_s \dot{\eta}_s + e_s \chi + \tfrac{1}{2} m_s v_s^2 + m_s U_s + m_s \psi) + g^2/8\pi G .$$

From the given Lagrangian formulation it is again possible to go over a Hamiltonian formalism similar to the one contained in the Section 3. In this case the Hamiltonian can be written as

$$H = \int \left\{ \sum_s \left[\frac{-1}{2 m_s \pi_{\Phi_s}} (\pi_{\Phi_s} \nabla \Phi_s + \pi_{\beta_s} \nabla \beta_s + \pi_{\eta_s} \nabla \eta_s - e_s \pi_{\Phi_s} \boldsymbol{A})^2 - m_s \pi_{\Phi_s} U_s (\pi_{\Phi_s}, \pi_{\eta_s}) - e_s \pi_{\Phi_s} \chi - m_s \pi_{\Phi_s} \psi \right] + \varepsilon_0 \left[\frac{c^2}{2} (\nabla \times \boldsymbol{A})^2 + \frac{1}{2\varepsilon_0^2} \sum_j \pi_{A_j}^2 \right] - \frac{g^2}{8\pi G} \right\} \mathrm{d}\boldsymbol{r} \quad (34)$$

or

$$H = \int \mathrm{d}\boldsymbol{r} \left\{ \varepsilon_0 \left(\frac{E^2}{2} + \frac{c^2}{2} B^2 \right) + \sum_s m_s n_s \left(\frac{v_s^2}{2} + U_s \right) + \frac{G}{2} \int \mathrm{d}\boldsymbol{r} \sum_s \frac{n_s(\boldsymbol{r}')\, n_s(\boldsymbol{r})}{|\boldsymbol{r} - \boldsymbol{r}'|} m_s^2 \right\}, \quad (35)$$

if the term $\nabla \cdot (\boldsymbol{g}\, \psi)$ is added to the integral appearing in (34) and if the constraint (31) is solved explicity.

The equations describing the model can now be derived by appropriately combining the corresponding Hamilton equations of motion. The fields and their conjugated momenta are the same as in the previous case.

5. Discussion

In the preceding sections the extended Clebsch representation introduced in [3] was applied for systems of equations describing ideal two-fluid plasmas. The internal energy per unit mass was assumed to be a function of the density and entropy. Variational principle were found, Hamiltonian formalisms were derived and canonical Poisson brackets were introduced.

The Hamiltonian densities obtained in terms of canonical Clebsch variables represent the total energy density. These may be applied to study nonlinear fluctuations in plasmas with finite pressure.

It seems to be difficult to find a variational principle for the system of equations describing two-fluid plasmas when finite gyroradii are considered. The term $-\nabla \cdot \boldsymbol{\pi}_s$ due to this effect [5] was added to eq. (1) and the following extended Clebsch representation was considered:

$$m_s v_{s_j} + e_s A_j = \frac{\partial \Phi_s}{\partial x_j} + S_s \frac{\partial \eta_s}{\partial x_j} + \sum_i \alpha_{s_i} \frac{\partial \beta_{s_i}}{\partial x_j}, \tag{36}$$

$$\boldsymbol{E} = -\dot{\boldsymbol{A}} - \nabla \chi, \quad \boldsymbol{B} = \nabla \times \boldsymbol{A}. \tag{37}$$

It was not possible to derive (2) and (4) from the Lagrangian densities considered. Even for electrostatic variations it was not possible to get the continuity (2). The dependence of $\boldsymbol{\pi}_s$ on $\boldsymbol{v}_s$ and $\boldsymbol{B}$ resulted in additional terms to (2) and (4) when the variations of η_s and $\boldsymbol{A}$ were considered. In the case considered before it was possible to write (1) in such a way that the vector potential only appeared through the combination

$$m_s \boldsymbol{v}_s + e_s \boldsymbol{A}.$$

Then, using the representation (9), it was possible to write this equation in terms of Φ_s, α_s, β_s, S_s and η_s only (see (10)). However, for the equation considered here the same combination does not appear in the new term $-\nabla \cdot \boldsymbol{\pi}_s$. This suggests the necessity of a different representation to obtain the variational principle for the model considered in this section, unless a constrained Lagrangian can be found as in the gravitational case.

A representation similar to (36) has been used to derive a variational principle for the equations of elasticity [3]. In this case $\boldsymbol{A} = 0$ and the stresses are just related to the internal energy.

After termination of our work we noticed in the literature [6] that a similar variational in the case of scalar pressure without gravitation has been found. In contrast to us [6] restricts to $\chi = 0$ gauge and ignores the problem of finite gyroradius.

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